Fair Shares, Mat

or Walk the

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Teaching young children to create equal-size groups is your treasure map for building students' flexible, connected understanding of and reasoning about ratios, fractions, and multiplicative operations.

Plank

Whether sharing a collection of toys among friends or a pie for dessert, children as young as kindergarten age are keen on making sure that everyone gets their "fair share." In the classroom, fair-sharing activities call for creating equal-size groups from a collection of objects or creating equal-size parts of a whole and are generally used by teachers to support students in formulating ideas of unit fractions. Yet most of us will readily acknowledge that as many students progress through the elementary school mathematics curriculum toward rational numbers, they have great difficulties in learning to reason about fractions, ratios, and multiplicative operations—all of which require reasoning about equal-size groups or parts. As teachers, how can we assist students in using their talent for and interest in fair sharing to overcome the challenges of learning rational numbers?

One of a child's greatest mathematical accomplishments is reasoning about rational numbers. A rich understanding of fractions, ratios, and multiplicative operations (multiplication, division, and scaling) requires gradual development and integration over the elementary and middle grades. Recognizing this, NCTM recommends in its Number and Operations Standard that "beyond understanding whole numbers, young children can be encouraged to understand and represent commonly used fractions in context … and to see fractions as part of a unit whole or of a collection" (NCTM 2000, p. 33). More recently, *Curriculum Focal Points* (NCTM 2006) prioritized the development of rational number reasoning (RNR) during the elementary school years.

Although proficiency in RNR is fundamental for success in higher mathematics, students have difficulties in coming to understand and use rational numbers (Lamon 2007). Teachers are familiar with these struggles, like the belief that "multiplication makes bigger, division makes smaller" or choosing to multiply or divide on the basis of the numbers in a problem rather than the problem context. However, young children *are* successful at creating equal-size groups or parts of collections and wholes, an idea that Confrey and colleagues refer to as *equipartitioning* (Confrey et al. 2009). In contrast to breaking a collection or whole into unequal-size groups or parts, equipartitioning describes children's cognitive ability to partition a set of objects or a whole into groups or parts of the same size.

In our work with children from prekindergarten through the middle grades, we sought to understand the ways that the children learned equipartitioning through fair-sharing activities and to chart these ways as they developed across the grades. Here, we present the different strategies, justifications, names, and mathematical relationships that students used as they engaged in fair-sharing activities (Confrey et al. 2010). From these observations, we offer an outline of the ways children use this early understanding to build successively more sophisticated ideas of rational number.

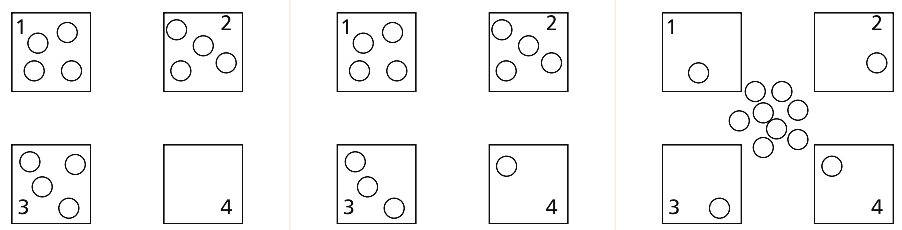
PIRATE: GLENDA POWERS/ISTOCKPHOTO.COM; TREASURE BORIS YANKOV/ISTOCKPHOTO.COM; X MARKS THE SPOT: ISTOCKPHOTO.COM

TABLE 1

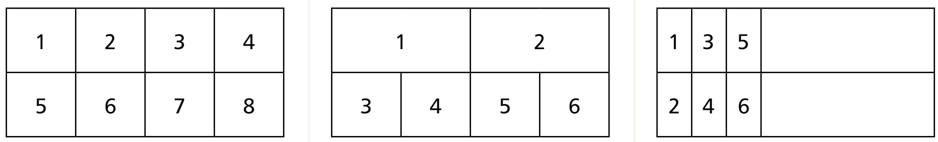
We hypothesize that children often have difficulty coordinating three different equipartitioning criteria as they create fair shares.

Sharing a collection of 12 objects among four students and a whole among six students when students are not coordinating the three criteria

Creating an incorrect number of parts from a collection	Creating unequal-size parts from a collection	Not exhausting the collection



Creating an incorrect number of parts from a whole	Creating unequal-size parts from a whole	Not exhausting the whole



Sharing collections

For young children, sharing a collection is a common experience, such as when sharing snacks or playing games. As early as preschool, children successfully share collections fairly—a success largely independent of counting skill (Pepper and Hunting 1998). They establish equivalence of shares using a one-to-one correspondence based on the repeated action of "dealing." We conjecture that as children create fair shares, they must coordinate three different equipartitioning criteria:

1. Creating the correct number of groups or parts
2. Generating equal-size groups or parts
3. Exhausting the collection or whole

Children have difficulties in coordinating these three criteria (see **table 1**).

Many opportunities exist for incorporating sharing collections into the normal activities of elementary school classrooms, such as snack time or distributing classroom materials. At other times, curricular topics can be adapted to include fair-sharing tasks, such as when writing number sentences. Teachers could read *The Doorbell Rang* (Hutchins 1989) and use the story as motivation to explore sharing collections fairly. In our work with children, we used a context of pirates finding treasure and attempting to fairly share it. The children received a collection of twenty-four gold coins and were asked, "How could you help these pirates fairly share the treasure?"

Sharing pirate treasure

The children used a number of strategies to fairly share the coins (see **table 2**). Less proficient students were unsuccessful at creating equal-size groups or exhausting the whole collection, instead focusing on creating a group for each pirate. When sharing between two pirates, very young children simply broke the collection into two piles, not attending to the number of coins in each. Others created shares of an equal number but did not use all the coins, demonstrating that they were not yet coordinating the three criteria. To assist students and prompt them to consider all the criteria, teachers could ask, "Which share of treasure would you rather have?" or "What about the leftover coins?"

Many successful students used a dealing strategy to solve the task. Although some students at first had a random approach, they quickly learned to deal systematically and use dealing as a means of efficiently creating equal-size groups. With more experiences, our students began to deal with larger composite units, such as dealing two or even three coins in early rounds and then switching to ones "because it is faster." Some of our older students first counted the coins in the treasure and used number facts to create fair shares. For example, when sharing between two pirates, one student recalled his doubles facts and reasoned, "Since twelve plus twelve equals twenty-four, each pirate gets twelve coins." Such connections are useful, but teachers should encourage these students to also make sense of their computations in the context of the original collection and the size of the shares.

When asked to explain how they knew that the shares were fair, children provided rich justifications. Some created lines of coins or stacked them, using early understanding of measurement. Many students counted to verify equivalence; others were satisfied that dealing was sufficient to create equal-size groups. Still others created arrays or visual patterns of coins to justify equivalence. Allowing students to share and defend their justification strategies with one another promotes all students' understanding and flexibility with these different approaches.

By encouraging students to name the results of sharing, teachers support them in creating foundations for different rational number conceptions that may develop from fair sharing. When sharing among three pirates, many students simply named the share as *eight*. Although this is a correct response, motivating students to think of additional ways of naming fair shares can prepare them for other conceptions of rational number. For instance, referring to the share as *one-third* fosters understanding rational numbers as a quantity and connects to early unit-fraction understanding. Referring to the share as *a third of the treasure* supports a view of rational number as an operator, which develops into ideas of scaling and multiplicative operations. Some children called a share *eight coins per pirate*, an early ratio understanding.

Sharing a whole

Fairly sharing a whole is also common in young children's lives, such as cutting sandwiches in quarters or slicing a pie. Young students are successful with these tasks, use a variety of strategies and explanations to "prove" that their shares are fair, and refer to shares in ways that can be cultivated to support flexible understanding of rational number.

Many opportunities exist for fair sharing to be incorporated into elementary school classrooms. During art activities, teachers could offer a group of students a large piece of paper to make a flag, with each student responsible for designing his or her fair share of it. During classroom parties, teachers could ask students to fairly share a giant cookie. They could read *Gator Pie* (Matthews 1995) and use it as motivation for sharing a whole. Curricular materials could be adapted for children to explore tasks of sharing a whole as well, such as exploring how lines of symmetry may create parts of equal size. In our work, we extended the pirate context from sharing a collection to include a birthday party for the pirates. We asked the children, "How could you help these pirates fairly share a birthday cake?"

Sharing pirate birthday cake

To share a whole, students must coordinate the size of the parts with the other two criteria. In this context, we observed several strategies that students used to share the cake (see **table 1**).

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Some unsuccessful students chopped the whole into various unequal parts and dealt pieces to each pirate. We concluded that by ignoring the size of the parts, they were simply interested in creating the correct number of shares. Others used a throw-away strategy, often by repeatedly halving to make shares, dealing one part to each pirate, and then arguing for the remaining shares to be discarded. To assist students, teachers may need to suggest not "wasting" any of the cake, or they might ask, "Which piece would you want?"

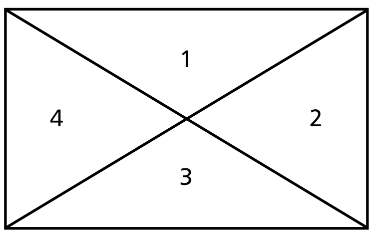
Most children successfully developed strategies for simpler tasks of sharing a whole (see **table 2**). Many were comfortable using halving and repeated-halving strategies. Many of them could easily cut a rectangle or circle in half and could also do so in multiple ways. Because halving is such an early action, children quickly move to a repeated-halving strategy, allowing them to fairly share for any power of two (e.g., four, eight, sixteen). As the number of pirates varied, many of the children developed a parallel-cutting strategy, making sequential cuts by "eye-balling" equal distances to create the desired number of parts. Although some were successful with this strategy, others experienced difficulties. For instance, one child made four parallel cuts to share for four people and asserted that he had shared fairly. When asked to show each of the four shares, he realized that he had actually created five shares. Through experiences like this, many students concluded that to produce n pieces, they must make $n - 1$ cuts.

Overall, fairly sharing round cakes was more difficult for the children than rectangular ones. They could halve or repeatedly halve a circle, but sharing among three and six was much more difficult. Whereas $n - 1$ horizontal or vertical cuts are sufficient for rectangles, cuts originating at the center, or *radial cuts*, are needed to create fair shares on a circle. Many of the unsuccessful students applied the parallel-cut strategy to the circle. Although most students struggled to make radial cuts, those who were successful made reference to their out-of-school experiences, such as seeing a peace sign. To justify fairness, some children used measurement ideas. Stacking was a common strategy observed across all grades and is similar to students' approaches to justifying with collections. After sharing among four pirates, one student stacked the four pieces on top of one another and explained that he was looking for "loose edges." If there were no "loose edges," each person got a fair share. Other children used geometric justification approaches. For instance, a kindergartner justified sharing a cake for two using symmetry: "OK, see, I folded the corners so that they touched—and all the sides matched. So that's how I knew." Such justifications afford opportunities for teachers to connect geometry and measurement to the development of number sense.

The ways that students name results when sharing a whole are similar to those when sharing a collection (see **table 2**). Some students stated counts, such as each pirate getting "one piece." Several students named the share as $1/n$th, suggesting an early understanding of part-whole fractions. Other students called a share "one *of* the n pieces" or "$1/n$th *of* the whole," signifying early understanding of rational number as a ratio or as an operator.

FIGURE 1

When sharing rectangular birthday cake among four pirates, some children claimed that triangles 2 and 3 were both fair shares because although triangle 3 is "longer and skinnier," triangle 2 is "shorter and fatter."



Mathematical relationships

As the children engaged in these fair-sharing tasks, they noticed patterns and developed ideas that teachers may recognize as the beginnings of more formal mathematical properties. These emergent relationships became tools that the children applied to new problems, and we believe they will eventually evolve into formal mathematical properties.

We discovered several such relationships, including compensation, composition, and continuity; but here we focus on compensation. When sharing collections, students began to qualitatively anticipate the relationship between the size of the share and the number of pirates sharing. That is, the more (or fewer) pirates there were to share the treasure, the fewer (or more) coins each pirate received. This relationship is important in measurement, and students may develop it into an understanding of inverse variation in the middle grades. Some students used this idea and reallocated shares among different numbers of pirates. For example, after sharing twenty-four coins among four pirates, one kindergarten student declared that if there were only three pirates, "each would get eight." She made this anticipation by taking one of the four shares, splitting it into three equal groups of two coins, and adding them to each of the remaining shares. We see this as an early form of the distributive property, where a quantity can be broken into parts of unequal size, and then each of those parts can be split into groups of equal size.

When sharing a whole, students began to believe that noncongruent parts could be of equal size, evidence of early ideas of compensation emerging (see **fig. 1**). Some older students used methods of decomposition to support their reasoning. One student decomposed each of the four triangles in half to create eighths and showed that triangles 1–4 were each composed of two of the congruent eighths. From these varied responses, we suggest that supplying opportunities to fairly share collections and wholes in multiple ways allows students to begin to formulate an understanding of mathematical relationships that will develop as they progress through the grades.

TABLE 2

A summary of activities shows that students used a number of strategies—for which they gave rich justifications—to share pirate treasure coins fairly.

	Sharing a collection	Sharing a whole
Strategies	Breaking	Chopping or throw away
	Dealing	Halving and repeated halving
	Dealing with composite units	Parallel and radial cutting
	Using number facts	Benchmarking
Justification	Stacking or lining up	Stacking
	Counting	Symmetry
	Arrays and visual patterns	
Naming	As a count	As a count
	As a quantity	As a quantity
	As an operator	As an operator
	As a ratio	As a ratio

A learning trajectory for equipartitioning

These findings are from a larger study articulating learning trajectories for rational number reasoning. Learning trajectories describe the ways in which naïve conceptions mature over time into powerful, connected mathematical ideas. The learning trajectory for equipartitioning (see **table 3**) explains how children use their informal knowledge of fair sharing as a resource to build an understanding of partitive division (Confrey in press). As described in the preceding sections, the trajectory begins with developing strategies for fairly sharing collections and a whole, represented on the bottom two levels of the table. As students experience these tasks, they learn and use the mathematical reasoning practices of justifying and naming (levels 3 and 4) and may notice emergent relationships such as compensation (levels 5–11) that ultimately come together as a generalization of partitive division. Influencing this progression are the different numbers of sharers and shapes of the wholes associated with the tasks, represented across the top of the table. After fairly sharing collections, students progress through creating equal-size parts of single wholes, first in halves and repeated halving, then by thirds, by even numbers, and eventually by odds. The upper levels of the trajectory address tasks that involve fairly sharing multiple wholes (Wilson et al. 2011).

TABLE 3

The learning trajectory for equipartitioning (Confrey in press) shows how children use informal knowledge of fair sharing as a resource to build an understanding of partitive division: $a \div b$ as a divided into b equal-size parts.

	Level	Description	Halving	Repeated halving (rectangles, then circles)	Thirds (rectangles, then circles)	Even splits (rectangles, then circles)	Odd splits (rectangles, then circles)
Upper levels of the Learning Trajectory for Equipartitioning describe the strategies and representations, mathematical reasoning practices, emergent relationships, and generalizations related to sharing multiple wholes.							
Emergent relationships	11	**Continuity principle**—a whole can be equipartitioned for all natural numbers greater than 1.					
	10	**Same splits of equal wholes are equal**—noncongruent parts resulting from the same split are equivalent.					
	9	**Reallocation**—extra shares can be reallocated for fewer people sharing collections.					
	8	**Quantitative compensation**—factor-based descriptions of the inverse relationship between the number of persons sharing and the size of a share					
	7	**Composition of splits**—splits can be composed to create nonprime outcomes.					
	6	**Qualitative compensation**—qualitative descriptions of the inverse relationship between the number of persons sharing and the size of a share					
	5	**Reassembly**—equal groups or parts can be recombined to produce the original collection or single whole as "*n* times as many" or "*n* times as much" as a single group or part.					
Mathematical practices	4	**Naming**—the shares resulting from equipartitioning collections or single wholes can be named in relation to the referent unit.					
	3	**Justification**—the equivalence of shares can be justified by counting, stacking, arrays, or patterns.					
Strategies	2	**Equipartition single wholes**—creation of equal-size parts of a single whole					
	1	**Equipartition collections**—creation of equal-size groups of a collection					

Conclusion

By providing instructional opportunities based on the general tendencies of students described by the learning trajectory, teachers can support students in developing strategies for equipartitioning through instructional tasks that assist students in coordinating the three criteria. By encouraging students to justify that all shares are of equal size and to mathematically name their shares, teachers can connect these fair-sharing experiences to other mathematical strands. These experiences may establish a cognitive foundation for more advanced rational number conceptions of fractions, ratios, and multiplicative operations. Finally, concentrated engagement in fair-sharing tasks allows children to notice quantitative relationships that may mature into formal mathematical properties later in their schooling.

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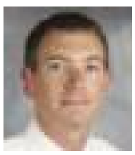


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