Supporting children to think flexibly about fractions

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Throughout all stages of primary school, learning fractions is frequently identified as one of the most difficult aspects of mathematics. One common issue with developing understanding of fractions, is that children are often provided with more experiences focused on the process relating to calculating and representing fractions, rather than experiences exploring the concepts of fractions. That is, children are not provided with various contexts, models and symbols where connections between whole number and fractional quantities can be made. In this article we describe the three fraction meanings that are critical for primary school students to explore and discuss how Confrey's (2012) mathematical practice of "naming" can help children develop flexible understandings of quantity.

Fractions are critical for mathematics

It is widely agreed that fractions are an important component of school curricula worldwide (Mullis et al., 2004), with studies such as Hilton et al. (2016) suggesting that children's understanding of fractions at a primary school level can predict their mathematical competency at high school and beyond. Yet fractions are frequently discussed as being particularly difficult to learn and to teach (Ball, 1993, Lamon, 2007, Misquitta, 2011). Many children experience difficulty establishing conceptual understandings of fractions, which includes the ability to read, recognise, compare and rename fractions in different contexts (Siemon, 2003). These contexts include exploring the different "personalities" of fractions, in that they can be used to represent part of a whole, a number, an operator, and a ratio.

There is no shortage of data that tells us children struggle with learning fractions and teachers struggle with teaching them. Mack (2001) and Ball (1993) infer that the way fractions are taught could hinder student learning. Their concerns stem from teaching approaches that not only focus on rote learning, or on symbols and algorithms over conceptual understanding (see also Smith, 2002), but approaches that emphasise the part-whole construct only. As discussed by Ball (1993), Mack (2001) and later by Siegler et al. (2013) and Gabriel (2016) students who have conceptual knowledge of fractions are better positioned to connect concepts and procedures. Most importantly though, this focus on conceptual understanding does not come at the expense of competence with use of symbols and algorithms. Both types of knowledge are interlinked and the development of both is an iterative process with no fixed order (Rittle-Johnson & Siegler, 1998).

Children's difficulties with making connections between fractions

One of the fundamental problems associated with poor fraction understanding is that children are not provided with opportunities that explore the different "personalities" or meanings of fractions which help children build connections between fractions and rational numbers more broadly. That is, they are generally exposed to the part-whole idea in which a whole object is divided into equal parts, and a number of these parts are selected (often by shading) and named as a fraction of the whole as a proper fraction only. For example, in a study conducted by O'Keeffe (2011) which asked upper primary students (equivalent to Year 6 and 7) to share their understandings of the concept of fraction, she noted the limited responses from each of the focus group cohorts. Their responses included comments such as "it's like a part of something", "a segment" (a student). Other, less confident responses included that it is a "piece of a whole number" (same comment made by three different students) and "a fraction is half of, is something, of a number". Ball (1993) highlights how a limited 'part-whole' view of fractions can lend itself to limited views of fractions at later stages. For example, when asked about the relationship between numbers and fractions, all students were confident that all numbers could be written as fractions however when probed about whether all fractions could be represented on a number line some of the students seemed to change their mind. Again, this aligns with the work of Lamon (1999), who highlights that students often struggle with the inconsistencies in regard to whole number operations and operations with fractions. For example, a student, initially said yes (that all numbers could be written as a fraction) until the follow up question was asked—"Is there any fraction that you don't think can be noted on a number line?" To this she responded, "Well ten tenths or something just be written as far as ten". With further probing the author tried to encourage the student to reason out what ten tenths would be equivalent to or could be represented as, however this was not successful. The child did however decide that you could expand your number line to accommodate 'ten tenths or eleven tenths'. We do not know the range of fractional representations these students have been exposed to, and hence we cannot make judgements of assumptions about this. However, the difficulties the children encountered when ask to share explain or give examples is typical of the limitations suggested by Siemon (2003) when children are not exposed to sharing their thinking and/or multiple representational models.

Three meanings of fractions

Confrey (2012) proposes that there are three meanings of fractions that encompass almost all of the fraction ideas and concepts students need to develop from the first year of primary school through to the middle years. These meanings are, 'fraction as a measure', 'fraction as operator' and 'fraction as a relation'.

Fraction as measure

The underpinning characteristic of ;fraction as a measure' meaning within the primary years considers a/b as a quantity that is defined in relation to a specified unit of 'one' but explores both proper fractions (part-whole idea) and improper fractions (composite units and unit fraction ideas). That means the quantity could be greater than one, because it lies between two whole units of that quantity. For example, comparing $\frac{3}{4}$ or $\frac{5}{3}$ within a fraction as measure meaning assumes they share same common referent unit of 1. Experiences that help promote this meaning of fractions include comparing fractions that can be represented on a number line.

Fraction as operator

'Fraction as an operator' refers to a multiplicative situation of partitioning—that is, fair shares are a result of equal parts or groups that are created from exhausting a whole. It includes ideas such as doubling and halving, times as many and 1 nth of..., which explore the multiplicative nature of increasing and decreasing quantities. In the early years of primary school, children can confidently explore these ideas with continuous, geometric contexts as well as set or whole number collections. In the upper primary years, children start to explore this meaning though exploring the relationship between division and multiplication on fractional quantities.

Fraction as relation

The fraction $\frac{a}{b}$ in this meaning can represent a ratio a:b if a and b share the same unit or a:b as a rate, whereby a and b are a comparison between different units. The basis of this meaning introduces children to the 'many-to-one' and 'proto-ratio' ideas. The many-to-one idea is the foundation to introducing simple ratio, in that it describes a relationship between two units that are related. For example, exploring the number of eggs required to make a quantity of muffins, can develop the many to one idea in the form of "one egg is required to make 6 muffins". Proto-ratio connects to the double and halving and times as many ideas from the fraction as operator meaning, that children experiment with the idea of simple ratio. For example, if one egg is 6 muffins, then 2 eggs is 12 muffins; or three times as many eggs means three times as many muffins.

To promote children's understanding and flexibility between the ideas, children need to explore the mathematical practice of naming and justifying quantities (Confrey, 2012). That is, they need experiences where they view quantities that involve whole number and fractions from multiple perspectives to build flexible understandings about whole number and fractional quantity. By promoting these ideas in the early years of schooling, children in the middle and upper primary years will be better equipped to deal with more complex ideas of ratio and proportion. The following activities demonstrate how a focus on naming and justifying quantity promotes flexibility between whole number and fractions ideas.

Making connections between the three meanings of fractions

The following activities were taken from a research project conducted by the first author in a study that explored how children make sense of the three meanings of fractions in the early years of primary school (Year 1 and 2). The first set of tasks were based on the picture book *The Doorbell Rang* by Pat Hutchins (1989). The children were asked to explore how to share 12 cookies between two, four and eight children. They were provided with paper cookies and plastic counters to explore. When children were creating their fair shares, they were asked to name the quantities in as many ways as possible. To support children in this process, the following prompts were used – described under each of the fraction meanings (column 1, Table 1). The key mathematical idea behind the prompts is outlined in column 2 (Table 1), while the types of responses children may provide that indicate these ideas are provided in column 3 (Table 1).

Table 1. Examples of the connection between the three meanings of fractions.

Fraction Meaning	Idea	Evidence of the idea
Fraction as Measure		
What do you notice about each share?	*Many-as-one*	Many cookies are one share
Can you describe a share?	*Composite unit*	Three cookies are a share
What fraction of the set is...?	*Unit fraction*	One quarter of the set is 3 cookies
Fraction as Operator		
How do you know each share is fair?	*Fair share*	Each person gets the same number of cookies/same size part of a cookie
Can you describe two shares as a fraction?	*1 nth of..*	Two shares are one half of the cookies
Can you describe three times as many shares?	*Times as many*	9 cookies is three times one share If I eat my cookies, there are three times as many fair shares left
Fraction as Relation		
Can you describe the connection between cookies and people?	*Many-to-one* *Proto ratio*	For each person there are three cookies 12 cookies for 3 people is the same as 24 cookies for 6 people

Children who have had little experience with fractions may start to explore the size of each share by folding parts, stacking and counting the cookies they are dealing with to check for equality. They may also deal one cookie at a time to exhaust the whole, in a trial-and-error manner to create equal groups which are all important ways of exploring quantity. However, to promote flexibility in understandings between whole number and fraction concepts, it is important that explicit opportunities for children to name the quantities from the different meanings are provided to lay the foundation for more flexible understandings of rational number knowledge to develop.

In a second set of learning experiences during this study, Year 1 and 2 children were asked to explore a range of fraction meanings in the context of maps. The students were provided with a range of carpet maps. One example is provided in Figure 1.

The children were asked to find the location of several dinosaurs who had "escaped an enclosure", based on clues that were provided, contextualised to each map. The following clues demonstrate the way in which children explore the connections between the three meanings of fractions in this activity:

Clue 1 (Fraction as measure focussed):

> Who walked further: a dinosaur that walked two thirds of the runway, or a dinosaur that walked three quarters? How do you know? Can you give an example on the map?

To support the fraction as measure meaning, children were also provided with strips of paper to scaffold their thinking (see Figure 2). Using the strips of paper, the children were asked to partition each strip into quarters and thirds, compare the different measures described in the clue and then apply this understanding to the pathways on the map.

This activity could be modified for older children, where they are provided with symbolic

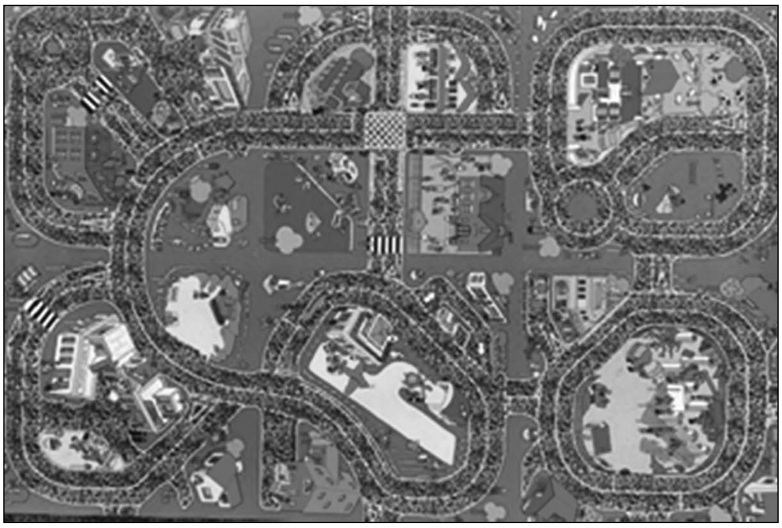

Figure 1. A carpet mat.

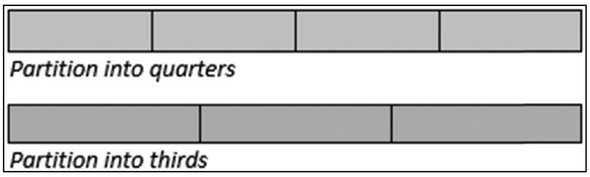

Figure 2. Example of paper strip pathways.

representations of more complex fractions within the clues (such as improper fractions), and they need to develop a strategy that will help them compare each measure.

Clue 2 (Fraction as operator focussed):

> If the length of the runway was half of the total distance a T-Rex walked, where would she be?

Again, children were provided with strips of paper to explore the language associated with doubling and halving, which is a core idea for establishing the Fraction as operator meaning. Eventually this was extended to the *times as many* idea and the children were prompted to connect this understanding back to the unit fraction/composite unit idea.

Clue 3 (Fraction as relation focussed):

> A dinosaur's footprint covers several white lines on the road. If nine lines were covered by the dinosaurs footprints, how many steps did she take? What is the pattern between the number of steps and the number of lines it travels?

The children would explore this idea by representing their thinking in typical drawings such as the following representation which demonstrated an appreciation of simple ratio (proto-ratio) idea (see Figure 3).

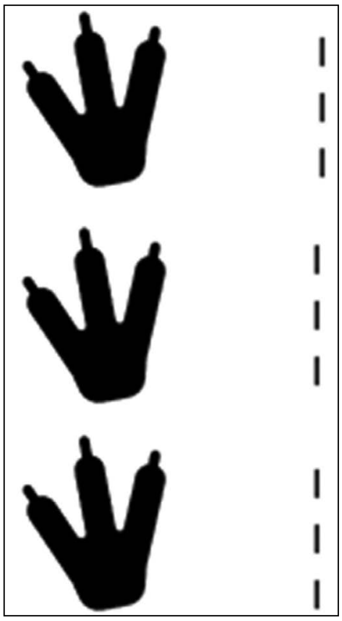

Figure 3. Example of children's representation of a ratio.

In this representation, the children demonstrated the relationship between the number of road lines and each footstep the dinosaur took, by presenting them in a column like format. Many children were able to name the relationship in this representation as: nine lines means there are three steps; there are three lines for each step.

Not only did the children work with complex ideas including fractions such as thirds, and fifths which are less easy to work with than halves and quarters, but they were able to view the different contexts through the three meanings of fractions that build fluency and flexibility between whole number and fractional quantity.

Conclusion

Given the importance of fractions to mathematical competency across school and beyond, it is important that children are able to develop and consolidate their understandings of fractions and whole number quantities in connected ways. This article puts forward an example of how Confrey (2012)'s three meanings of fractions can be embedded into learning activities to help children develop flexible understandings about early rational number ideas.

Not only are the example activities a way of showing the interconnection between multiplication, division, fraction, and proportional ideas, but they are also appropriate for children to explore across all year levels in primary school. When children are provided with explicit experiences to explore these connections, fractions not only make sense but children's rational number development in general is promoted. Therefore, children's conceptual understating of how numbers and quantities can be named and viewed before introducing symbolic notation can be developed, providing a far richer foundation for students in the primary years to engage in this critical mathematical domain.

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