A LEARNING TRAJECTORY FOR EQUIPARTITIONING

Equipartioning can be characterized as follows:

- Equipartitioning (or *splitting*) indicates cognitive behaviors that have the goal of producing equal-size groups (from collections) or pieces (from continuous wholes) as "fair shares" for each of a set of individuals.
- Equipartitioning is *not* breaking, fracturing, fragmenting, or segmenting in which there is the creation of unequal parts.
- Equipartitioning is the foundation of division and multiplication, ratio, rate, and fraction (Confrey et al., 2009).

Originally, equipartitioning was not one of the learning trajectories we planned for rational number reasoning; those included five concept areas: (1) multiplication and division; (2) length and area; (3) fraction, ratio, and rate; (4) decimals and percent; and (5) similarity and scaling. However, based on the map of rational number reasoning (Figure 3.5), we determined first to identify research related to the construct of splitting, or partitioning, as this had been previously identified as a fundamental construct for multiplicative reasoning (Confrey, 1988). In reviewing the literature, and based on prior work on splitting (Confrey, 1988, 1994; Confrey & Scarano, 1995), we recognized that a substantial body of knowledge supported a key role for fair sharing in young children (Confrey, Maloney, Nguyen et al., 2008; Confrey et al., 2009). This literature began with studies of young children sharing collections (Hunting & Sharpley, 1991; Pepper, 1991) and a whole (Confrey et al., 2009; Empson & Turner, 2006; Pothier & Sawada, 1983) but also included studies of older children sharing multiple wholes (Charles & Nason, 2000; Lamon, 1996; Toluk & Middleton, 2003). The body of research spanned investigations into a variety of topics in early mathematics, including partitioning itself, one-to-one correspondence and counting, and fractions. However, no one had conducted a fair-sharing case-based analysis of this literature, which Confrey undertook based on her conjecture that the concept of splitting developed in parallel (independently, but intertwined) with that of counting. This belief led to a proposal of three cases for analysis as the basis for rational number reasoning: equipartitioning collections (Case A), single wholes (Case B), and multiple wholes to produce a proper fraction or an improper fraction/mixed number (Case C).[1]

COGNITIVE ELEMENTS OF THE EQUIPARTITIONING LEARNING TRAJECTORY: A FRAMEWORK FOR UNDERSTANDING

From the analysis of clinical interviews, a framework for the cognitive elements was identified that could be applied across the cases. It employed a parallel structure to capture what students have learned within and among the cases. This framework of cognitive elements facilitates capturing the process of movement within and between the proficiency levels. The development of this combination of processes accounts for the generative learning progress through the proficiency levels, a key to the construction of learning trajectories.

An underlying structure for students' accomplishments emerged, described as a "framework for understanding" (Confrey, Maloney, Wilson, & Nguyen, 2010):

1. Strategies used to solve the problems
2. Mathematical reasoning practices used to explain the strategies and solutions, including naming the results of tasks and justifying them
3. Emergent mathematical relations or properties, which act as local generalizations to guide future approaches
4. Systematic tendencies toward certain errors or misconceptions (and their resolutions)
5. Broader generalizations of increasing mathematical power.

Below, these five cognitive elements are illustrated, with excerpts from clinical interviews.

Strategies

Rayna, a second-grade student, was asked to share a pile of coins ("pirates' treasure") fairly between two pirates. Her strategy was to separate the coins into two piles by dealing one coin to one pirate, then one coin to the other pirate, round by round until all the coins were distributed. She was not completely systematic, however: While she dealt one coin to each pirate on each round, sometimes the first coin of the round went to the left pile, and sometimes the first coin went to the right pile. Nonetheless, when she was asked how she knew each pirates has a fair share of the treasure, she placed two coins in the middle of the table and explained, "If you have one on this side [sliding one coin to her right] and one on this side [sliding the second coin to her left], it's even because each of them has one." Rayna's strategy accomplished the goal of the challenge, and she explained her strategy in ways that were consistent with behavior observed in the interview, namely, dealing in one-to-one correspondence from the original pile of coins to each of the recipient piles.

Mathematical Reasoning Practices

The following excerpt demonstrates the development of the mathematical practice of justification in the context of reallocating pirate treasure to ensure a fair share for the remaining pirates when one of the pirates departs (uncharacteristically leaving his share of the treasure behind). Emma, age 5, had just successfully shared 24 coins among four pirates, producing 6 coins per pirate, arranged in 2-by-3 arrays (Figure 3.6a). She was then asked if she could share the whole treasure fairly among three pirates—if one pirate sailed away without his treasure. She thought briefly, then collected all the coins into a single large pile on the table in front of her (Figure 3.6b)

Student: Okay. How much cents are there when three pirates do not have any money? Zero cents! [She then assembles a 2-by-4 array of coins for one pirate, without audibly counting, and an identical array for the second pirate. As she completes the second array, she says] I'm using eight. [She then creates a row of 5 coins and a row of 3 coins for the third pirate, but looks a little perplexed.]

S: Oh no! He [pointing to the row of 3 coins for the third pirate], he has less … Wait. [counts the two rows, which has 5 coins in one row and 3 in the other, and moves one coin from the longer row over to the shorter row, producing a third 2-by-4 array] They all have 8. Eight was the magic number!

Interviewer: How did you know they each got the same amount?

S: Okay. Last time it was 6 [touches one or two rows of the array of 8, and rests index finger of each hand on the top row of the array] and I just added 2 more because he [pointing to where the original fourth pirate's pile of 6 coins had been] had 6, and I added 2 more to each one [sweeping her hand quickly over the other pirates' piles] which makes 6. One, two … three, four … five, six [tapping in turn the two coins at the far end of each of the 3 arrays] … so I thought that *eight* was the magic number (Figure 3.6c).

This example illustrates what we believe is a step in the evolution of children's understanding of compensation, from qualitative to quantitative compensation. Qualitative compensation refers to children's recognition or belief that if more (or fewer) people share a quantity (a collection of objects, or a single object that can be shared) than had initially shared it,

FIGURE 3.6a. Emma has shared 24 coins fairly among 4 pirates

FIGURE 3.6b. After one pirate left the island, Emma collected all 24 coins into a single pile

FIGURE 3.6c. Emma has constructed three 2-by-4 arrays of coins

FIGURE 3.6. Equipartitioning and reallocation with justification, a mathematical reasoning practice

then each person's share will be less (or more). By comparison, *quantitative compensation* encompasses children's abilities to predict, demonstrate, explain, and justify with specific quantitative arguments, the relationship between changes in the number of persons sharing and the changes in the sizes of the shares. In this episode, Emma reassembled the entire collection, as though she were going to deal the three pirates' shares anew, but then directly constructed three 2-by-4-coin arrays, one for each of the three remaining pirates, in order. Then she explains that 8 was the "magic number," explaining that she got the eight by adding two coins to each of the original shares, and furthermore, that the two coins for each of the pirates came from the original 6 coins from the pirate who departed: "I just added 2 more because he had 6 [indicating with her hand the pirate who left the island] [and] I added 2 more, which makes 6," making the direct connection between the 6 coins of the share of the pirate who left, and two more coins for each of the remaining three pirates, to share all six of those coins evenly among the other three pirates.

Emergent Relations and Properties

Emergent properties are exemplified by students becoming aware of a relationship within or property of mathematical structure that can assist them in solving more complex tasks. This example illustrates how the mathematical practice of justification becomes incorporated into other levels of proficiencies along the learning trajectory. Evan, a second grader, had been asked to share a play-dough rectangle fairly for four people. Stating that he was " just gonna do this the hardest way there is," he cut the rectangle along both diagonals (Figure 3.7a), comparing this to his depiction of the way "most people" would think of doing it (i.e., making parallel cuts, as shown in Figures 3.7b).

Of the four triangles that result from the diagonal cuts, the opposite pairs of triangles are congruent, but adjacent pairs of triangles are not. We have observed repeatedly that children often initially believe that fair shares must be congruent. Even very young children readily recognize that a rectangular cake can be shared fairly between two persons by cutting it across the diagonal, because the resulting pieces are "the same." If asked to share the same rectangle fairly for four persons, many children will draw the second diagonal, but when asked how they know the four parts are fair shares, they often reply that the four parts are *not* the same. Many students will then argue that the pieces are fair (i.e., equal) because two of the triangles are shorter and fatter while the other two are taller and skinnier; this is a qualitative compensation-based justification. Evan, however, made an argument that approached an informal geometric proof. He formed a parallelogram from each pair of opposite triangles by adjoining them along the original rectangle's (opposite) sides, as in Figures 3.8b and 3.8c. Evan

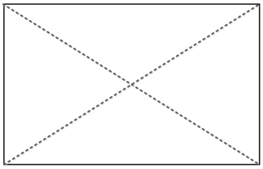

FIGURE 3.7a. Evan's diagonal split of a cake into four pieces

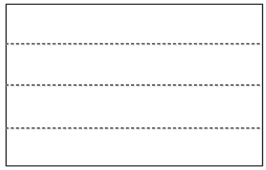

FIGURE 3.7b. Evan's split of a cake into four parallel pieces

FIGURE 3.7. Equipartitioning a rectangular cake for four persons

stated in turn that the parallelograms (Figures 3.8b and 3.8c) each had two fair shares.

Then he placed triangles *C* and triangle *D* next to each other, and argued (qualitative compensation) that *C* was taller, and *D* was shorter and fatter, and that they were the same. Evan's argument had some resemblance to the transitive property of equality for numerical situations: the parallelograms are the same, they are each split in two equally parts, and therefore the halves of each parallelogram are equal to each other. In similar ways, at a relatively sophisticated level, students will assert the equivalence of two parts based on the fact that if two areas are congruent and they are equipartitioned into the same number of parts, the parts must be equivalent; we refer to this emergent relationship as the *property of equality of equipartitioning*, or *PEEQ*. It facilitates children solving a wide array of tasks in which equivalence of shares does not depend on geometric congruence of the shares. A second, less complex example of PEEQ is displayed in the top proficiency level of the assessment item shown in Figure 11.

Such emergent properties go beyond the development of strategies or mathematical reasoning practices. They appear to function for students

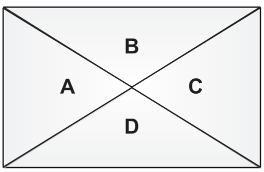

FIGURE 3.8a. Evan's diagonal 4-split with pieces marked for ease of recognition

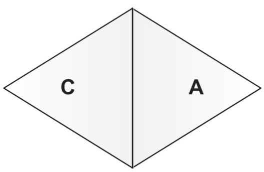

FIGURE 3.8b. Evan's right and left triangles joined to form a parallelogram

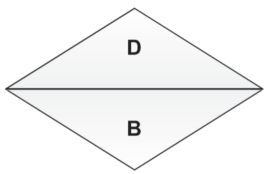

FIGURE 3.8c. Evan's upper and lower triangles joined to form a congruent parallelogram

FIGURE 3.8. Property of equality of equipartitioning: Evan's demonstration of the equivalence of the four diagonally cut shares

as "local" generalizations that permit them to approach a broad range of tasks effectively and efficiently. They often incorporate initial strategies and methods of justification into more general principles or conjectures that are useful in capturing the underlying structures in the mathematics.

Misconceptions

As illustrated previously in the conceptual corridor (Figure 3.1), the cognitive elements of mathematical reasoning also include obstacles that students are likely to encounter and which they must resolve in order to lay the groundwork for later acquisition of more advanced concepts. One key obstacle, an additive misconception in the context of orthogonal splits, commonly surfaces when children explore a variety of ways to equipartition rectangles. It is common for children to initially assume that the total number of parts produced from splitting into m parts along one side (for instance, splitting the rectangle vertically into m parts) and then along the perpendicular side with an n-split (horizontally into n parts) will be $m + n$. Coming to understand that the total number of parts is actually $m \times n$—because each of the horizontal cuts acts on *all* of the m vertically cut parts to produce $m \times n$ parts (or vice versa)—represents a fundamental means of resolving the additive misconceptions.

A first-grader, Kate, was asked to find a way to share a rectangle among six persons other than by making all horizontal or all vertical parallel cuts. On her first attempt, she made a vertical 3-split and a horizontal 3-split; that is, she cut the rectangular play-dough cake into three equal parts vertically and three equal parts horizontally (Figure 3.9a). Upon counting the parts, she realized that this made nine parts instead of six. At first she saw no way to resolve the problem, but with encouragement tried again. She split the rectangle horizontally into three parts again, then made a vertical cut about one third of the way across. She scored it lightly about two thirds of the way across the rectangle, and prepared to split it again vertically (Figure 3.9b). Had she continued, she would again have made nine pieces. Then she paused and considered this work intently. She used her fingers to seal ("erase") the first cut, and to smooth out the score mark two thirds of the way across the rectangle, and then simply cut the cake in half vertically. Counting to verify she had obtaining the desired six parts (Figure 3.9c), she sat back smiling. She had effectively figured out how to split the first (horizontal) 3-split by a second (vertical) 2-split to produce *six* fair shares. This splitting of splits is called *composition of splits*.[2] It represents a key step toward the roots of multiplication.

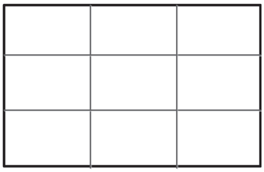

FIGURE 3.9a. Kate's first result of attempting sharing a rectangle fairly for 6 without making parallel cuts

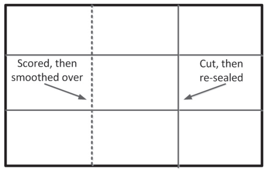

FIGURE 3.9b. Kate's second try at sharing the rectangle for 6

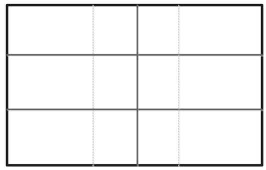

FIGURE 3.9c. Finally, on the third try, a 2-by-3 six-split composed successfully

FIGURE 3.9. Composition of splits

Generalization

In an interview involving Case C equipartitioning (i.e., multiple whole dissectible items shared among multiple persons) and a setting in which the number of items was fewer than the number of people sharing, fifth grader Ellie explored several tasks. One of these was to share two pizzas (construction-paper circles) among three persons.

She first equipartitioned each circle into 3 parts by drawing three radii on each circle, then cut the thirds apart (Figure 3.10a). Then she dealt the parts of the circles into three piles of two pieces each, saying, "There we go … They each get … all of these are thirds of one whole pizza. So they're each getting [for each pair of pieces, she places the two pieces adjacent to each other along matching sides, to form two thirds of a single pizza] two thirds of one whole pizza. But, [each person's share is] a third, wait, we know, yes, a *third*, of *two* pizzas" (Figure 3.10b).

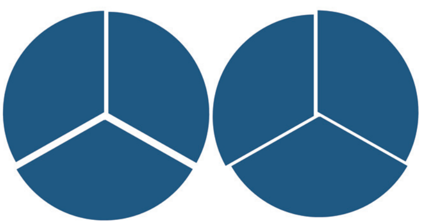

FIGURE 3.10a. Ellie's mark-all strategy: split each whole into three parts, one for each of the three people sharing

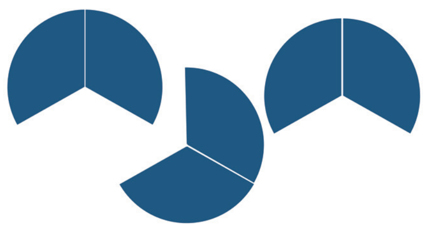

FIGURE 3.10b. After dealing the six pieces into three equal (fair) shares

FIGURE 3.10. Generalization of equipartitioning a items for b people

Ellie's strategy is one known as *mark-all* (Lamon, 1996). She recognized that for multiple wholes, even though her strategy for making a set of fair shares involves splitting each whole into the same number of parts as people, the final share per person is nonetheless represented by the unit fraction $1/n$ (n = the number of persons) of the original collection of pizzas. But her generalization also recognizes two equivalent ways of answering, depending on the unit to which she refers (i.e., the "referent unit"): Each person's share is *one third of the original total* amount of pizza, as well as *two thirds of a single* pizza.

REPRESENTING PROFICIENCY LEVELS AND TASK CLASSES

Delineating the variety of cognitive reasoning elements in our framework, combined with the three cases, led us to propose the current form of our learning trajectory. The learning trajectory for equipartitioning is proving to be foundational for the development of the whole rational number reasoning space under construction. We argue that it is foundational because through our research we are demonstrating how the roots of division and multiplication sit within the trajectory, with key links to geometric ideas of length, area, similarity, and scaling. Further, we are demonstrating in related work the key linkages between this work and the development of fractions, ratio relations, and the construct of a/b-as-operator. We claim that these three key ideas form the primary structures necessary for a robust understanding of rational number reasoning (Confrey, Maloney, Nguyen et al., 2008; Confrey et al., 2009).

A vertical display of learning trajectory proficiency levels was developed to incorporate the three cases of equipartitioning, as well as precise statements of the knowledge and skills that we conjectured should accrue to students through successfully solving case-specific challenges. Table 3.1 delineates the 16 proficiency levels[3] of the equipartitioning learning trajectory, listed from less sophisticated at the bottom to more sophisticated at the top.

For each proficiency level the outcome space, an ordered list (also ordered from least to most sophisticated) combined student reasoning results we had observed in the clinical interviews and predictions of those we expected as we expanded the tasks into assessment items. The outcome spaces are designed (again, iteratively) to convey sufficient detail to understand the cognitive behaviors associated with the proficiency level. They served to guide the development of paper-and-pencil assessment items.

To ensure consideration of student understanding across the range of equipartitioning cases and across the K through 7 grade levels, a two-dimensional display of proficiency levels and task types was added. The proficiency levels in one dimension were arrayed with a second dimension known as *task classes* (Table 3.2).

REFERENCES

Almond, R. G., Mislevy, R. J., Steinberg, L. S., Williamson, D. M., & Yan, D. (in press). *Bayesian networks in educational assessment.* New York, NY: Springer.

Battista, M. T. (2009, August). *Cognition Based Assessment.* Paper presented at the Realizing the Potential for Learning Trajectories Research to Serve as Evidence and Validation for Standards and Related Assessments Conference, Raleigh, NC.

Behr, M. J., Cramer, K., Harel, G., Lesh, R., & Post, T. (2011). *The rational number project.* Retrieved from http://www.cehd.umn.edu/rationalnumberproject/

Charles, K., & Nason, R. (2000). Young children's partitioning strategies. *Educational Studies in Mathematics, 43*(2), 191–221.

Clements, D. H., & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning, 6*(2), 81–89.

Clements, D. H., Wilson, D. C., & Sarama, J. (2004). Young children's composition of geometric figures: A learning trajectory. *Mathematical Thinking and Learning, 6*(2), 163–184.

Confrey, J. (1988). Multiplication and splitting: Their role in understanding exponential functions. In M. Behr, C. LaCompagne, & M. Wheeler (Eds.), *Proceedings of the Tenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 250–259). DeKalb, IL: Northern Illinois University Press.

Confrey, J. (1994). Splitting, similarity, and rate of change: a new approach to multiplication and exponential functions. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 293–332). Albany, NY: State University of New York Press.

Confrey, J. (2006). The evolution of design studies as methodology. In R. K. Sawyer (Ed.), *The Cambridge handbook of the learning sciences* (pp. 135–152). New York, NY: Cambridge University Press.

Confrey, J. (2008, July). *A synthesis of the research on rational number reasoning: A learning progressions approach to synthesis.* Paper presented at the 11th International Congress of Mathematics Instruction, Monterrey, Mexico.

Confrey, J. (2012). Better measurement of higher-cognitive processes through learning trajectories and diagnostic assessments in mathematics: The challenge in adolescence. In V. Reyna, S. Chapman, M. Dougherty, & J. Confrey (Eds.), *The adolescent brain: Learning, reasoning, and decision making* (pp. 155–182). Washington, DC: American Psychological Association.

Confrey, J., & Maloney, A. P. (2012). Next generation digital classroom assessment based on learning trajectories in mathematics. In C. Dede & J. Richards (Eds.), *Steps toward a digital teaching platform* (pp. 134–152). New York, NY: Teachers College Press.

Confrey, J., Maloney, A., & Nguyen, K. H. (2008). Rational number reasoning database. Retrieved from http://gismo.fi.ncsu.edu/index.php?module=Search

Confrey, J., Maloney, A., Nguyen, K., Mojica, G., & Myers, M. (2009). Equipartitioning/splitting as a foundation of rational number reasoning using learning trajectories. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics*

Education (Vol. 2, pp. 345–352). Thessaloniki, Greece: International Group for the Psychology of Mathematics Education.

Confrey, J., Maloney, A., Nguyen, K., Wilson, P. H., & Mojica, G. (2008, April). *Synthesizing research on rational number reasoning*. Paper presented at the Research Pre-Session of the Annual Meeting of the National Council of Teachers of Mathematics. Salt Lake City, UT.

Confrey, J., Maloney, A. P., Wilson, H., & Nguyen, K. (2010, May). *Understanding over time: The cognitive underpinnings of learning trajectories*. Paper presented at the Annual Meeting of the American Educational Research Association, Denver, CO.

Confrey, J., Rupp, A. A., Maloney, A. P., & Nguyen, K. (unpublished). Toward developing and representing a novel learning trajectory for equipartitioning: Literature review, diagnostic assessment design, and explanatory item response.

Confrey, J., & Scarano, G. H. (1995, October). *Splitting reexamined: Results from a three-year longitudinal study of children in grades three to five*. Paper presented at the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Columbus, OH.

Cooper, H. (1998). *Synthesizing research*. Thousand Oaks, CA: Sage.

deBoeck, P., & Wilson, M. (Eds.). (2004). *Explanatory item response models: A generalized linear and nonlinear approach*. New York, NY: Springer.

Desimone, L. M., & Le Floch, K. C. (2004). Are we asking the right questions? Using cognitive interviews to improve surveys in education research. *Educational Evaluation and Policy Analysis, 26*(1), 1–22.

Elmore, R. F. (2002). *Bridging the gap between standards and achievement*. Washington, DC: Albert Shanker Institute.

Empson, S. B., & Turner, E. (2006). The emergence of multiplicative thinking in children's solutions to paper-folding tasks. *Journal of Mathematical Behavior, 25*, 46–56.

Frey, A., Hartig, J., & Rupp, A. A. (2009). An NCME instructional module on booklet designs in large-scale assessments of student achievement: Theory and practice. *Educational Measurement: Issues and Practice, 28*(3), 39–53.

Ginsburg, H. P., Kossan, N. E., Schwartz, R., & Swanson, D. (1983). Protocol methods in research on mathematical thinking. In H. P. Ginsburg (Ed.), *The Development of Mathematical Thinking* (pp. 7–47). New York, NY: Academic Press.

Hunting, R. P., & Sharpley, C. F. (1991). Pre-fraction concepts of preschoolers. In R. P. Hunting & G. Davis (Eds.), *Early fraction learning* (pp. 9–26). New York, NY: Springer-Verlag.

Kane, M. T. (2006). Content-related validity evidence in test development. In M. S. Downing & M. T. Haladyna (Eds.), *Handbook of test development* (pp. 131–153). Mahwah, NJ: Lawrence Erlbaum Associates.

Lamon, S. J. (1996). The development of unitizing: Its role in children's partitioning strategies. *Journal for Research in Mathematics Education, 27*(2), 170–193.

Maloney, A. P., & Confrey, J. (2010, July). *The construction, refinement, and early validation of the equipartitioning learning trajectory*. Paper presented at the 9th International Conference of the Learning Sciences, Chicago, IL.

Mislevy, R. J., Steinberg, L. S., & Almond, R. A. (2003). On the structure of educational assessments. *Measurement: Interdisciplinary Research and Perspectives, 1*, 3–67.

Mislevy, R. J., Steinberg, L. S., Almond, R. A., & Lukas, J. F. (2006). Concepts, terminology, and basic models of evidence-centered design. In D. M. Williamson, I. I. Bejar, & R. J. Mislevy (Eds.), *Automated scoring of complex tasks in computer-based testing* (pp. 15–48). Mahwah, NJ: Erlbaum.

Mojica, G. (2010). *Preparing pre-service elementary teachers to teach mathematics with learning trajectories.* Unpublished doctoral dissertation, North Carolina State University, Raleigh, NC.

Opper, S. (1977). Piaget's clinical method. *Journal of children's Mathematical Behavior, 5*, 90–107.

Pellegrino, J. W., Chudowsky, N., & Glaser, R. (2001). *Knowing what students know: The science and design of educational assessment.* Washington, DC: National Academy Press.

Pepper, K. L. (1991). Preschoolers' knowledge of counting and sharing in discrete quantity settings. In R. Hunting & G. Davis (Eds.), *Early fraction learning* (pp. 103–129). Heidelberg, Germany: Springer-Verlag.

Piaget, J. (1970). *Genetic epistemology.* New York, NY: W.W. Norton.

Pothier, Y., & Sawada, D. (1983). Partitioning: The emergence of rational number ideas in young children. *Journal for Research in Mathematics Education, 14*(4), 307–317.

Rupp, A. A., & Templin, J. L. (2008). Unique characteristics of diagnostic classification models: A comprehensive review of the current state-of-the-art. *Measurement, 6*, 219–262.

Rupp, A. A., Templin, J. L., & Henson, R. (2010). *Diagnostic measurement: Theory, methods, and applications.* New York, NY: Guilford Press.

Rutkowski, L., Gonzalez, E., Joncas, M., & von Davier, M. (2010). International large-scale assessment data: Issues in secondary analyses and reporting. *Educational Researcher, 39*, 142–151.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114–145.

Toluk, Z., & Middleton, J. A. (2003). The development of children's understanding of the quotient: A teaching experiment. Arizona State University: *Unpublished.*

Wilson, M. (2005). *Constructing measures: An item response modeling approach.* Mahwah, NJ: Lawrence Erlbaum Associates.

Wilson, P. H. (2009). *Teachers' uses of a learning trajectory for equipartitioning.* Unpublished doctoral thesis, North Carolina State University, Raleigh, NC.

Yilmaz, Z. 2011. *Toward understanding of students' strategies on reallocation and co-variation items: In relation to an equipartitioning learning trajectory.* Unpublished master's thesis, North Carolina State University, Raleigh, NC.